Hyperbolically Monotone Densities

Lennart Bondesson[1]
[1] Umeå University, Sweden

We look at probability density functions $f(x)$ on $\mathbf{R}_+ = (0, \infty)$ such that, for each $u > 0$, the function, $h(w) = f(uv)f(u/v)$ as a function of $w = v+v^{-1}$ is decreasing. The class of such pdfs is called the class of hyperbolically monotone (HM) densities on $\mathbf{R}_+$. It turns out that $X \sim$ HM if and only if $\log X$ has a logconcave, i.e. strongly unimodal pdf. A $U(a, b)$ distribution has a density which is HM.

Requiring more generally, for a fixed integer $k \geq 1$, that

$$(-1)^\nu h^{(\nu)}(w) \downarrow \quad \text{for} \quad \nu = 0, 1, \ldots, k-1,$$

we get the HM_k class of densities. The HM_k classes were introduced in Bondesson (1992, pp 103-104) and further studied in Bondesson (1997). Obviously HM_1=HM. For $k = \infty$ we get the HM_∞ class which also is called the class of hyperbolically completely monotone (HCM) densities. Clearly

$$\text{HCM} \subset \ldots \subset \text{HM}_2 \subset \text{HM}_1.$$

Surprisingly many well-known standard distributions are in the HCM-class. For instance, the gamma and the lognormal densities are HCM.

The HM_k classes have several interesting properties. For example, each such class is closed wrt to multiplication of independent random variables:

$$X \sim \text{HM}_k, Y \sim \text{HM}_k \;\Rightarrow\; XY \sim\; \text{HM}_k.$$

For $k = 1$, this result corresponds to the well-known result that the class of logconcave densities on $\mathbf{R}$ is closed wrt convolution, cf. Ibragimov (1956). Moreover, $X \sim \text{HM}_k \Rightarrow X^q \sim \text{HM}_k$ for $|q| \geq 1$.

In part this talk is motivated by a recent result by Roynette, Vallois & Yor (2009) which is here generalized to the general result that

$$X \sim \text{Gamma}(k, 1),\; Y \sim \text{HM}_k \;\; (X, Y \,\text{independent}) \;\Rightarrow\; XY \sim\; \text{GGC},$$

where GGC is the class of generalized gamma convolutions. GGC = all limits of convolutions of gamma distributions (see Bondesson, 1992, or Steutel & van Harn, 2004). The GGC class has recently been shown to be closed with respect to multiplication of independent random variables (Bondesson, 2014).

Open problems are mentioned.

Keywords: Generalized gamma convolution, Hyperbolic monotonicity, Logconcavity.

References:

Bondesson, L. (1992). *Generalized Gamma Convolutions and Related Classes of Distributions and Densities.* Lecture Notes in Statistics *76*. New York: Springer.

Bondesson, L. (1997). On hyperbolically monotone densities. In: N.L. Johnson and N. Balakrishnan (Eds), *Advances in the Theory and Practice of Statistics: A Volume in Honor of S. Kotz* (pp. 299-313). New York: John Wiley & Sons.

Bondesson, L. (2014). A class of probability distributions that is closed with respect to addition as well as multiplication of independent random variables. To appear in *J. Theoret. Probab.* DOI 10.1007/s10959-013-0523-y

Ibragimov, I.A. (1956). On the composition of unimodal distributions. *Theor. Probab. Appl. 1*, 255-260.

Roynette, B., P. Vallois, M. Yor (2009). A family of generalized gamma convoluted variables. *Probab. Math. Statist. 29*, 181-204.

Steutel, F.W., K. van Harn (2004). *Infinite Divisibility of Probability Distributions on the Real Line.* New York: Marcel Dekker.