Stochastic processes associated with cross-diffusion parabolic systems

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We derive a stochastic model associated with a fully nondiagonal system of parabolic equations

$$\begin{cases} u^1_t = [u^1(u^1+u^2)_x]_x + (1-u^1-u^2)u^1, \quad u^1(0,x) = u^1_0(x) \\ u^2_t = [u^2(u^1+u^2)_x]_x + \gamma(1-u^1-\kappa u^2)u^2, \quad u^2(0,x) = u^2_0(x), \end{cases} \tag{1}$$

that arises as a mathematical model of cell growth under inhibition [1]. To this end we consider a system of stochastic equations and prove that their solution allow to construct a probabilistic representation of a generalized solution to the Cauchy problem (1).

Let $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}$, $t \ge 0, x \in R$, $\mathcal{H}^1_T = \{u \in L^2([0,T]\times R; R) : u_x \in L^2([0,T]\times R; R)\}$, $\mathcal{D}$ be the Schwartz space of test functions and $\langle u, h\rangle = \int_R h(x)u(x)dx$.

We say that pair of scalar functions $u^q \in \mathcal{H}^1_T \times \in \mathcal{H}^1_T, q = 1,2$, is a weak ((regular weak) solution of (1), provided $u^q_x \in L^2_{\rm loc}([0,T]\times R)$, ($u^q_x \in L^2_{\rm loc}([0,T]\times R) \cap C^{1,2}([0,T]\times R)$) and for arbitrary test functions $h^q(t) \in \mathcal{H}^1([0,T];\mathcal{D})$

$$\langle u^1(t), h^1(t)\rangle - \int_0^t \langle u^1(\theta), h^1_\theta(\theta)\rangle d\theta + \int_0^t \langle [u^1_x(\theta) + u^2_x(\theta)], h^1_x(\theta)\rangle\, d\theta$$

$$= \langle u^1(0), h^1(0)\rangle + \int_0^t \langle u^1(\theta)(1-u^1(\theta)-u^2(\theta)), h^1(\theta)\rangle d\theta, \tag{2}$$

$$\langle u^2(t), h^2(t)\rangle - \int_0^t \langle u^2(\theta), h^2_\theta(\theta)\rangle d\theta + \int_0^t \langle [u^2_x(\theta) + u^1_x(\theta,x)], h^2(\theta)\rangle\, d\theta$$

$$= \langle u^2(0), h^2(0)\rangle + \int_0^t \langle u^2(\theta)\gamma(1-u^1(\theta)-\kappa u^2(\theta)), h^2(\theta)\rangle\, d\theta. \tag{3}$$

Given a probability space $(\Omega, \mathcal{F}, P)$ and standard independent Wiener processes $w^q(t) \in R$, we consider stochastic equations for $0 \le t \le \theta \le T < \infty$, $q = 1,2$,

$$d\xi^q(\theta) = -m^q_{\{u^1,u^2\}}(\theta, \xi^q(\theta))d\theta - M_{\{u^1,u^2\}}(\theta,\xi^q(\theta))dw^q(\theta), \quad \xi^q(t) = x, \tag{4}$$

where $m^1_{\{u^1,u^2\}}(t,x) = (u^1(t,x)+u^2(t,x))\frac{u^1_x(t,x)}{u^1(t,x)}$, $m^2_{\{u^1,u^2\}}(t,x) = (u^1(x) + u^2(x))\frac{u^2_x(t,x)}{u^2(t,x)}$ and $M_{\{u^1,u^2\}}(t,x) = \sqrt{2(u^1(t,x)+u^2(t,x))}$.

Let u^q, $q=1,2$, be regular weak solutions of (1), with $u_0^q \in C^{2+\epsilon}$, $\epsilon>0$, and $u_0^q \geq \varpi > 0$. Under this assumption coefficients in (4) satisfy conditions of the classical existence and uniqueness of SDE solutions and hence there exist $\xi_{t,x}^q(\theta)$ satisfying to (4).

Given $f_1 = 1-u^1-u^2, \quad f_2=\gamma(1-u^1-\kappa u^2)$, by the Feynman -Kac formula we obtain

$$h^1(t,x) = E\left[\exp\left\{\int_t^T f_1(u^1(\theta,\xi_{t,x}^1(\theta)),u^2(\theta,\xi_{t,x}^1(\theta)))d\theta\right\} h_0^1(\xi_{t,x}^1(T))\right],$$

$$h^2(t,x) = E\left[\exp\left\{\int_t^T f_2(u^1(\theta,\xi_{t,x}^2(\theta)),u^2(\theta,\xi_{t,x}^2(\theta)))d\theta\right\} h_0^2(\xi_{t,x}^2(T))\right]$$

define the classical solution of the Cauchy problem

$$h_\theta^1+[u^1+u^2]h_{xx}^1+(u^1+u^2)\frac{u_x^1}{u^1}h_x^1+(1-u^1-u^2)h^1=0, \quad h^1(T,x)=h_0^1(x) \quad (5)$$

$$h_\theta^2+[u^1+u^2]h_{xx}^2+(u^1+u^2)\frac{u_x^2}{u^2}h_x^2+\gamma(1-u^1-\kappa u^2)h^2=0 \quad h^2(T,x)=h_0^2(x), \quad (6)$$

provided h_0^1, h_0^2 are smooth and bounded. Our main result is the following.

Theorem. *Assume that $u_0^1, u_0^2 \in \mathcal{H}^1 \cap C^2$ and there exists a unique regular weak solution (u^1,u^2) of the Cauchy problem (1) such that $u^1(t), u^2(t) \in \mathcal{H}_T^1 \cap C^2(R^d)$. Let the processes $\xi^1(t)$,$\xi^2(t)$ satisfy (4), while $\hat\xi^1(t)$, $\hat\xi^2(t)$ be the correspondent time reversal processes (see [2]). Then, for any test functions h^1, h^2 functions*

$$u^q(t,x) = E_{t,x}\left[\exp\left\{\int_0^t f_q(\hat\xi^q(\theta))d\theta\right\} u_0^q(\hat\xi^q(t))\right], \quad q=1,2, \quad (7)$$

satisfy integral identities (2), (3)and hence give a probabilistic representation of a weak solution (u^1,u^2) to (1).

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