# Probabilistic schemes of the Cauchy problem numerical solution for quasilinear parabolic equations

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The aim of this talk is to derive numerical schemes based on probabilistic representations of the Cauchy problem solution for a quasilinear parabolic equation of the form

$$u_s + \langle a, \nabla \rangle u + \frac{1}{2} TrA\nabla^2 uA + g(x, u, \nabla u) = 0, \quad u(T, x) = u_0(x), \qquad (1)$$

where $a, A$ have the form $a = a(x, u, \nabla u)$ and $A = A(x, u, \nabla u)$, $TrA\nabla^2 uA^* = A_i^k \nabla_{ij} u A_j^k$, $\langle a, \nabla \rangle u = \sum_{k=1}^d a_k \frac{\partial u}{\partial x_k}$. Actually the corresponding schemes can be derived for different types of the Cauchy problem solutions, namely for classical, generalized and viscosity solutions, depending on what kind of probabilistic counterpart of (1) is chosen (see [1]-[3]).

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and $w(t) \in R^d$ be a standard Wiener process defined on it. For a semilinear parabolic equation of the type (1) when functions $f = a, A, g$ are of the form $f = f(x, u)$ one can consider the following stochastic counterparts for (1)

$$d\xi(t) = a(\xi(t), u(t, \xi(t)))dt + A(\xi(t), u(t, \xi(t)))dw(t), \quad \xi(s) = x \qquad (2)$$

$$u(s, x) = E[u_0(\xi(T)) + \int_s^T g(\xi(\theta), u(\theta, \xi(\theta)))d\theta] \qquad (3)$$

in terms of forward SDEs and

$$d\xi(\theta) = a(\xi(\theta), y(\theta))d\theta + A(\xi(\theta), y(\theta))dw(\theta), \quad \xi(s) = x, \qquad (4)$$

$$dy(\theta) = -g(\xi(\theta), y(\theta))dt + z(\theta)dw(\theta), \quad y(T) = u_0(\xi(T)). \qquad (5)$$

Here we assume $y(\theta) = u(\theta, \xi(\theta))$, $z(\theta) = A^* \nabla u(\theta, \xi(\theta))$. The extension of (2),(3) to a quasilinear case is based on a construction of probabilistic representations of $\nabla u$ and perhaps $\nabla^2 u$ and deriving of a system similar to (2),(3) for a function $M(s, x) = (u(s, x), \nabla u(s, x), \nabla^2 u(s, x)) \in R \times R^d \times R^d \otimes R^d$. At the other case (4),(5) need only minor changes and have the form

$$d\xi(\theta) = a(\xi(\theta), y(\theta), z(\theta))d\theta + A(\xi(\theta), y(\theta), z(\theta))dw(\theta), \quad \xi(s) = x, \qquad (6)$$

$$dy(\theta) = -g(\xi(\theta), y(\theta), z(\theta))dt + z(\theta)dw(\theta), \quad y(T) = u_0(\xi(T)). \qquad (7)$$

To obtain numerical algorithms to solve (1) one can consider a partition of an interval [0,T], $0 = t_0 \leq \ldots \leq t_n = T$, where a unique solution of (1) is proved to exist in a certain sense and consider the corresponding discrete versions of either (2),(3) or (4),(5) or else (6),(7) (see [4],[5]).

As an example we consider the discrete version of (4),(5) (with $g$ depending on $z$) written in the form

$$\xi_0^n = x, \quad \bar{\xi}_{k+1} = \bar{\xi}_k + a(\bar{\xi}_k, \bar{y})h + A(\bar{\xi}_k, \bar{y})\epsilon\sqrt{h},$$

$$\bar{y}_n^n = u_0(\bar{\xi}_n^n), \quad \bar{z}_k^n = h^{-1}E_{t_k}[\bar{y}_{k+1}^n\sqrt{h}\epsilon]$$

$$\bar{y}_k^n = E_{t_k}[\bar{y}_{k+1}^n - g(\bar{\xi}_k^n, \bar{y}_k^n, \bar{z}_k^n)h].$$

Here $\epsilon$ is chosen to be either a standard normal variable $N(0,1)$ or a binomial variable valued in $\{1,-1\}$ with $P\{\epsilon = 1\} = P\{\epsilon = -1\} = 1/2$.

The corresponding scheme with $\epsilon$ being an $N(0,1)$ r.v. was realized to obtain a numerical solution for the Burgers equation with various initial data.

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