Scale space multiresolution analysis of time series

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Most time series contain features at various different time scales, i.e. they have fine local detail at short time scales and smooth average detail at long time scales. The purpose of a recently developed method (Pasanen et al. 2013) is to extract these scale dependent details from time series data. Consider as an example the time series $\mathbf{y}$ shown in Figure 1. The example series $\mathbf{y}$ is a sum of three sine components $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_3$ and a Gaussian noise component. Our aim is to extract these components.

Denote by $\mathbf{S}_\lambda$ a smoothing operator with a smoothing level $\lambda \geq 0$, for which $\mathbf{S}_0\mathbf{y} = \mathbf{y}$, and $\mathbf{S}_\lambda\mathbf{y}$ converges to the linear regression line, as $\lambda \to \infty$. We employ here a discrete spline smoother (see e.g. Pasanen et al. 2013) but the method is applicable to other such smoothers as well. The scale dependent components are extracted using a smoothing level sequence $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_L \leq \infty$ as follows:

$$\mathbf{y} = \sum_{i=1}^{L-1}(\mathbf{S}_{\lambda_i} - \mathbf{S}_{\lambda_{i+1}})\mathbf{y} + \mathbf{S}_{\lambda_L}\mathbf{y} - \bar{\mathbf{y}}\mathbf{1} + \bar{\mathbf{y}}\mathbf{1} \equiv \sum_{i=1}^{L+1}\mathbf{z}_i, \tag{1}$$

where $\mathbf{z}_i = (\mathbf{S}_{\lambda_i} - \mathbf{S}_{\lambda_{i+1}})\mathbf{y}$ for $i = 1, \ldots, L-1$, $\mathbf{z}_L = \mathbf{S}_L\mathbf{y} - \bar{\mathbf{y}}\mathbf{1}$ and $\mathbf{z}_{L+1} = \bar{\mathbf{y}}\mathbf{1}$, where $\bar{\mathbf{y}}$ is the mean of $\mathbf{y}$. The scale dependent component $\mathbf{z}_i$ can be interpreted as the detail lost when smoothing is increased from λ_i to λ_{i+1}. If $\lambda_L = \infty$, $\mathbf{z}_L$ corresponds to the linear trend.

To achieve a good separation of the scale-dependent components, the smoothing level sequence in (1) needs to be chosen carefully. A principled method for choosing this sequence can be based on the derivative of the smooth $\mathbf{S}_\lambda\mathbf{y}$ with respect to $\log\lambda$. This derivative is visualized using a so called scale-derivative map. The scale-derivative map of $\mathbf{y}$ is shown in the left panel of Figure 2. The horizontal axis of the map relates to time points and the vertical axis relates to the smoothing parameter value. The depth of the color signifies the value of the derivative at a certain time point and smoothing level indicated by the horizontal and vertical axes. The three sines and the noise component come up in the map as four oscillating bands of white and black. The correct smoothing parameter sequence is found at the intersections of these bands where the norm of the derivative has a local minimum, marked by black lines. The extracted components (omitting the mean) are presented in the right panel of Figure 2. The first component, shown in the topmost panel, captures the noise and the three other components seem to estimate the corresponding sine waves reasonably well.

Because the observed series $\mathbf{y}$ contains noise, the extracted components $\mathbf{z}_i$, $i > 1$, are also corrupted and statistical inference is thus needed to establish the credibility of their features. For this, we utilize Bayesian inference by first modeling the posterior distribution of the noiseless time series $\boldsymbol{\mu}$. Then, instead of finding the components $\mathbf{z}_i$ from $\mathbf{y}$, we perform the decomposition (1) on the posterior mean of $\boldsymbol{\mu}$. Finally, the time intervals where the components $\mathbf{z}_i$ are credibly positive or negative are inferred.

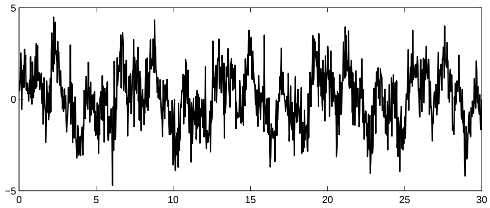

Figure 1: The time series $\mathbf{y}$ that consists of a sum of three sine waves and Gaussian noise.

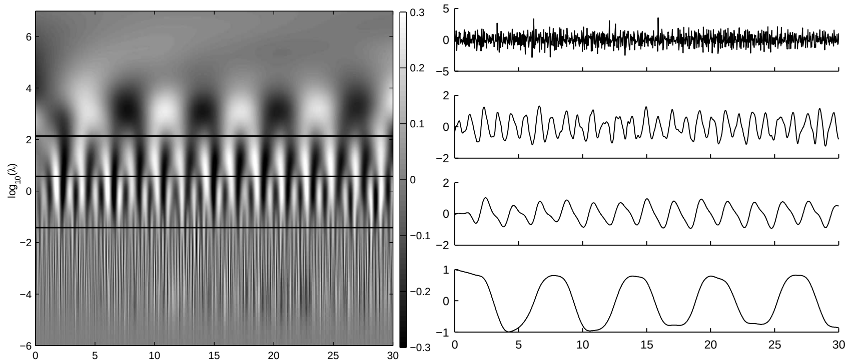

Figure 2: The left panel shows the scale-derivative map of $\mathbf{y}$. The black lines indicate the local minima of the norm of the derivative. The right panel shows the components $\mathbf{z}_1, \ldots, \mathbf{z}_4$ obtained by computing differences of smooths of $\mathbf{y}$ using the smoothing levels marked by the black lines in the left panel.

Keywords: Scale space, Time series, Visualization, Bayesian methods

References:

Pasanen, L., Launonen, I. and Holmström, L. (2013), A scale space multiresolution method for extraction of time series features. *Stat*, *2*: 273-291.