On an optimal variance stopping problem

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Problem formulation

Let X_t be a regular non-killed linear diffusion defined on a filtered probability space $(\Omega, \mathcal{F}, F, \mathbf{P})$ evolving on an interval $(0, \infty)$. For simplicity, let the boundaries 0 and ∞ be natural.

We study a variance stopping problem

$$V(x) := \sup_{\tau} \mathbf{Var}_x \{X_\tau\} = \sup_{\tau} \mathbf{E}_x \left\{ (X_\tau - \mathbf{E}_x \{X_\tau\})^2 \right\}, \tag{1}$$

where $\mathbf{E}_x$ is the expectation conditioned on the initial state $X_0 = x$, and the supremum is taken over all $\mathcal{F}$-stopping times.

Motivation

By maximizing the variance, we get a tight upper bound for the variance of a process over all admissible stopping times. In finance, for example, one is usually concerned about the variation of an investment or a portfolio. Hence the maximal variance can be seen as a measure of risk: The variation of an investment strategy cannot exceed the maximal value for any admissible stopping time.

In a larger scene, the variance stopping problem can be seen as a first step towards more generally defined optimal stopping problems. The next posts will be optimal stopping of skewness and kurtosis, or optimal stopping of the variance with respect to some reward function. These all are non-linear problems with respect to the expectation and consequently very non-trivial problems.

Results

The main difficulty in the variance stopping problem (1) is the fact that the greatly developed machinery applied for solving usual optimal stopping problems is not readily usable. This is due to the non-linearity of the variance problem. In Pedersen (2011) this obstacle is overcome by linking a certain usual stopping problem to the variance stopping problem (see Theorem 2.1 in Pedersen (2011)). He then goes on to solve some explicit examples using this connection.

We will exploit this connection more deeply applying a measure change method from Beibel and Lerche (2000). We will show that under some weak conditions the solution is finite and can be written explicitly. Moreover, we will find sufficient conditions under which the value (1) is infinite. We will see that the scale function of the diffusion is the main character in our show.

More explicitly, we will show that if the diffusion grows "too fast", then the variation is infinite. Moreover, if the diffusion does not grow "too fast" and the function $b\frac{S'(b)}{S(b)}$, where $S(x)$ is the scale function, is increasing, then the value is finite and we can immediately write it down explicitly.

Keywords: Linear diffusion, optimal stopping, variance.

References:

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