# Some Inequalities on the Spherically Truncated Multinormal Distribution

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Let $X(1), \ldots, X(v)$ be independent, normally distributed random variables with zero means and respective variances $a(1), \ldots, a(v)$. Inspired by Palombi and Toti (2013), hereafter referred to as PT, we consider their truncated distribution over the interior of a $v$-dimensional sphere centered at the origin. With reference to this truncated distribution, where $X(1), \ldots, X(v)$ are no longer independent, we explore the inequalities

$$\mathrm{Var}\{Y(n)\} \leq 2a(n)\mathrm{E}\{Y(n)\}, \quad \text{for every } n,$$

and

$$\mathrm{Cov}\{Y(n), Y(m)\} \leq 0, \quad \text{for every } n \neq m,$$

where $Y(n)$ is the square of $X(n)$. These will be called the variance and covariance inequalities, respectively.

As noted in PT, the main interest in these inequalities originates from the fact that, if universally true, they are necessary and sufficient for the convergence of a fixed-point algorithm in Palombi, Toti and Filippini (2012) for the reconstruction of $a(1), \ldots, a(v)$ in case the only available information amounts to the covariance matrix of $X(1), \ldots, X(v)$ in the truncated setup mentioned above. Such reconstruction can be of practical importance, for example, in compositional analysis of multivariate log-normal data affected by outlying contaminations. Another motivation for these inequalities arises from non-linear optimization issues. We refer to PT for further details.

A complete proof of these inequalities is, however, quite nontrivial because, in general, exact calculation of moments is not possible in the truncated setup. An informative and in-depth discussion of the variance inequality was recently given in PT, where its truth was established in some special cases. A proof in full generality was left as an open challenge. The covariance inequality was not discussed in PT and this also remains open.

The present paper aims at completely proving the two inequalities. The concept of monotone likelihood ratio is useful in our proofs. Moreover, we also prove and utilize the fact that the cumulative distribution function of any positive linear combination of independent chi-square variates is log-concave, even though the same may not be true for the corresponding density function. This log-concavity result, which is new to the best of our knowledge, should be of independent interest.

**Keywords:** Chi-square distribution, Log-concavity, Monotone likelihood ratio.

## References:


Palombi, F., S. Toti (2013). A note on the variance of the square components of a normal multivariate within a Euclidean ball. *J. Multivariate Anal. 122*, 355–376.

Palombi, F., S. Toti, R. Filippini (2012). Numerical reconstruction of the covariance matrix of a spherically truncated multinormal distribution. ArXiv e-prints http://arxiv.org/abs/1202.1838.