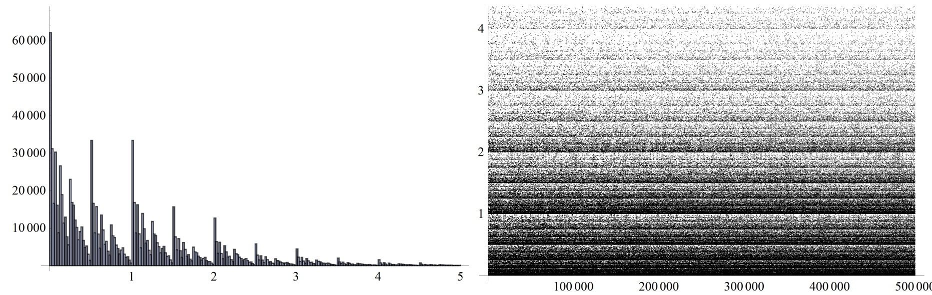
Figure 1: Plot of the Markov chain when $p = 1/3$

(a) Histogram (b) Empirical mass distribution



The fine structure of the stationary distribution for a simple Markov chain

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We define a Markov chain by

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ \frac{1}{2}X_n & \text{with probability } 1-p. \end{cases}$$

We are interested in analyzing the fractal properties of its stationary distribution.

Put into the context of iterated function systems (IFS), the process may be described as an IFS on $\mathbb{R}$ with two maps, $w_1(x) = x + 1$ and $w_2(x) = x/2$, and associated probability vector $(p, 1-p)$. This IFS doesn't have a bounded attractor since w_1 is not a contraction. However, the IFS still satisfies the so-called average contractivity condition and thus the stationary distribution satisfying

$$\mu = p w_1^{-1} + (1-p) w_2^{-1}$$

exists and is unique.

As the figure suggests, the distribution exhibits typical fractal properties (numerical analysis suggests that this occurs when $p \le 1/2$). We are consequently investigating the fractal (Hausdorff) dimension of the distribution, with the goal to understand its multifractal spectrum.

This example is of interest since it does not satisfy the so-called open set condition, which is often assumed in order to limit the overlapping of the maps in an IFS, greatly simplifying the geometry of the process. There exist some results when overlap occurs, most of them concerned with strict

contractivity and bounded attractors. In the case of average contractivity, only upper bounds for the dimension are known.

Keywords: Iterated function systems, Hausdorff dimension, fractal

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