A Gaussian Mixture Autoregressive Model

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During the past few decades various nonlinear autoregressive (AR) models have been proposed to model time series data. A general class of nonlinear AR models is comprised of mixtures of linear AR models studied in works by Wong and Li (2001), Glasbey (2001), Lanne and Saikkonen (2003), Carvalho and Tanner (2005), Dueker, Sola, and Spagnolo (2007), and Bec, Rahbek, and Shephard (2008) to mention only a few.

For a real-valued time series y_t $(t = 1, 2, \ldots)$, a mixture AR model can be defined by assuming that the conditional distribution of y_t given its past history $\mathcal{F}_{t-1} = \{y_{t-j},\ j > 0\}$ is a mixture of distributions. In the Gaussian case, this amounts to assuming that the conditional density function is of the form

$$f(y_t \mid \mathcal{F}_{t-1}) = \sum_{m=1}^{M} \alpha_{m,t} \frac{1}{\sigma_m} \phi\left(\frac{y_t - \mu_{m,t}}{\sigma_m}\right).$$

Here the (positive) mixing weights $\alpha_{m,t}$ are $\mathcal{F}_{t-1}$–measurable and satisfy $\sum_{m=1}^{M} \alpha_{m,t} = 1$, $\phi(\cdot)$ signifies the density function of a standard normal random variable, $\mu_{m,t}$ is defined as

$$\mu_{m,t} = \varphi_{m,0} + \sum_{i=1}^{p} \varphi_{m,i} y_{t-i}, \quad m = 1, \ldots, M,$$

and $\boldsymbol{\vartheta}_m = (\varphi_{m,0}, \varphi_m, \sigma_m^2)$, where $\varphi_m = (\varphi_{m,1}, \ldots, \varphi_{m,p})$ and $\sigma_m^2 > 0$, contain the unknown parameters introduced in the above equations. In the special case $M = 1$, the model reduces to a conventional linear Gaussian AR model with conditional mean $\mu_{1,t}$ and conditional variance σ_1^2. When $M > 1$, a general class of mixtures of such models is obtained, and particular members of this class are defined by different specifications of the mixing weights $\alpha_{m,t}$.

The Gaussian mixture autoregressive model we consider extends the model proposed by Glasbey (2001) in the first order case $p = 1$. We assume that the AR parameters φ_m $(m = 1, ..., M)$ satisfy the usual stationarity condition of the linear AR(p) model and define the mixing weights as

$$\alpha_{m,t} = \frac{\alpha_m \mathsf{n}_p(\boldsymbol{y}_{t-1}; \boldsymbol{\vartheta}_m)}{\sum_{n=1}^{M} \alpha_n \mathsf{n}_p(\boldsymbol{y}_{t-1}; \boldsymbol{\vartheta}_n)},$$

where the $\alpha_m \in (0, 1)$ are unknown parameters satisfying $\sum_{m=1}^{M} \alpha_m = 1$, $\boldsymbol{y}_{t-1} = (y_{t-1}, \ldots, y_{t-p})$, and $\mathsf{n}_p(\cdot\,; \boldsymbol{\vartheta}_m)$ signifies the density function of the p–dimensional normal distribution with mean and covariance matrix defined by the mean and covariance matrix of the vector $\boldsymbol{\nu}_{m,t-1} = (\nu_{m,t-1}, \ldots, \nu_{m,t-p})$, where $\nu_{m,t}$ is an auxiliary stationary linear Gaussian AR(p) process $\nu_{m,t} = \varphi_{m,0} + \sum_{i=1}^{p} \varphi_{m,i} \nu_{m,t-i} + \sigma_m \varepsilon_t,\ \varepsilon_t \sim NID(0, 1)$ $(m = 1, \ldots, M)$. In addition to extending the model proposed by Glasbey (2001) we also provide a more thorough discussion of its theoretical properties.

Our model differs from its previous nonlinear alternatives in several advantageous ways. A major theoretical advantage is that conditions for stationarity and ergodicity are always met and these properties are much more straightforward to establish than is common in nonlinear autoregressive models. Another major advantage is that explicit expressions of the stationary

distributions of dimension $p+1$ or smaller are known and given by mixtures of Gaussian distributions with constant mixing weights $\alpha_1, ..., \alpha_M$. Due to the known stationary distribution exact maximum likelihood estimation is feasible, and one can assess the applicability of the model in advance by using a nonparametric estimate of the stationary density. An empirical example with interest rate series is used to illustrate the practical usefulness and flexibility of the model, particularly in allowing for level shifts and temporary changes in variance. The paper is available at http://blogs.helsinki.fi/saikkone/files/2014/04/GMAR-R1.pdf.

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