Pathwise stochastic integrals and Itô formula for multidimensional Gaussian processes

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Let $X = (X^1, \ldots, X^n)$ be n-dimensional Gaussian process on $[0, T]$ with all X^k independent. We are interested in which generality stochastic integrals $\sum_{k=1}^n \int_0^T f(X_u^1, \ldots, X_u^n) dX_u^k$ exist in pathwise sense. We do not assume processes X^k are semimartingales, and hence standard integration techniques can not be applied.

Young showed that if $f(x)$ has finite p variation and $g(x)$ has finite q variation with $\frac{1}{p} + \frac{1}{q} > 1$, then $\int f(x)dg(x)$ exists as a limit of Riemann-Stieltjes sums. Föllmer introduced pathwise forward-type Riemann-Stieltjes integration and Zähle introduced the generalized Lebesgue-Stieltjes integrals, and later on Nualart and Răşcanu further developed the theory. The case with fractional Brownian motion was studied by Azmoodeh et al. to show that if B^H is a fractional Brownian motion with $H > \frac{1}{2}$ and f is a convex function with left-sided derivative f'_-, then $\int_0^T f'_-(B_s^H) dB_s^H$ exists as a generalized Lebesgue-Stieltjes integral. Later on, Sottinen and Viitasaari generalized the fractional Brownian motion case to a large class of Gaussian processes with some mild extra assumptions. They showed that the result also holds for Gaussian processes with α-Hölder continuous trajectories for some $\alpha > \frac{1}{2}$. However all mentioned studies above only considered one dimensional case.

In this talk we study the existence of pathwise stochastic integrals with respect to a general class of n-dimensional Gaussian processes. Moreover, we consider a wide class of adapted integrands that are functions of locally bounded variation with respect to all variables. With some mild integrability assumption, we can prove the existence of integral $\int_0^T f(X_u^1, \ldots, X_u^n) dX_u^k$ for every $k = 1, \ldots, n$ in the sense of generalized Lebesgue-Stieltjes integral provided that each Gaussian process belongs to certain class and $f(x_1, \ldots, x_n)$ is of locally bounded variation separately with respect to each variable. Furthermore, it is not even necessory that processes X^k are independent provided that conditional processes belong to the considered class. We also derive the multidimensional Itô formula. Some extensions and applications will be discussed.

Keywords: Föllmer integral, generalized Lebesgue-Stieltjes integral, Itô formula, pathwise stochastic integral.

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